A characterization of the function $\pi(x)$ and a demonstration of the twin primes conjecture

By

Marco Bortolamasi[*]

Abstract

A specific property of the function $\pi(x)$ *is provided, resulting in a solution to the twin primes conjecture.*

A new characterization of twin primes is provided, constituting one of the few criteria available in literature.

The result lends itself well to processing further applications and insights.

Key words: *twin primes, characterization, function* $\pi(x)$*, twin primes conjecture.*

[*] Order of Engineers of the province of Modena, c/o Department of Engineering E. Ferrari, University of Modena and Reggio Emilia, P. Vivarelli 10, 41125 MODENA; e-mail: bortolamasim@libero.it.

Notation

In addition to the symbols commonly used:

$\left\lfloor \frac{x}{y} \right\rfloor$ = floor function of $\frac{x}{y}$, y≠0

P = set of primes

$I°(n) = \{i \ : \ \text{odd, with } 3 \leq i \leq n\}$,

$\left\{ \frac{x}{y} \right\}$ = fractional part function of $\frac{x}{y}$.

Introduction

Two primes are *twin primes* if their difference is 2. It's still a conjecture the Euclid's statement about the existence of infinitely many twin primes. Conditions are known instead, in order to prove that a pair (p-2, p) is a pair of twin primes.

In 1949 /**3**/ P.A. Clement demonstrated that *integers n, n+2 are a pair of twin primes if and only if:*

$$4\,[(n-1)! + 1] \equiv -n \pmod{n(n+2)}$$

In 1963 /**8**/ F. Pellegrino demonstrated the following theorem, deriving from Wilson's theorem:

Two natural numbers p-2 e p, with p ≥ 5, are both primes if and only if:

$$4\left[\frac{(p-3)!}{p-2}\right] \equiv -5 \quad (mod.\, p)$$

In 2004 S.M. Ruiz /**11**/ demonstrated that:

For odd n > 7, the pair (p, p+2) of integers are twin primes if and only if:

$$\sum_{i\ odd}^{j}\left(\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor + \left\lfloor\frac{p}{i}\right\rfloor - \left\lfloor\frac{p-1}{i}\right\rfloor\right) = 2$$

where the summation is over odd values of i through $j = \lfloor p/3 \rfloor$

In this paper is presented an interesting property of the function π(x) enabling to prove that the pairs of twin primes are infinitely many and so giving a solution to the ancient conjecture of the twin primes.

Now we establish one lemma which will become useful in proving Theorem 1.

Lemma1

Let p ∈ P then p+2 ∈ P if and only if:

$$\sum_i \left(\left\lfloor \frac{p+2}{i} \right\rfloor - \left\lfloor \frac{p+1}{i} \right\rfloor \right) = 0 \; con \;\; i \in I°(p) \qquad (1)$$

<u>*Proof*</u>

If p+2 ∈ P then for all natural numbers *i*: $1 < i \leq p$:

$$\left\lfloor \frac{p+2}{i} \right\rfloor - \left\lfloor \frac{p+1}{i} \right\rfloor = 0 \qquad (2)$$

In fact it is well known that: $\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor = 0$ if i don't divide n ∈ N, n > 0

with $i \in N \;\; i \neq 1 \;\; i < n$

Hence any prime verifies (2) and consequently the statement because any term of the summation (1) is greater than or equal to zero.

At the same time:

If $\left\lfloor \frac{p+2}{i} \right\rfloor - \left\lfloor \frac{p+1}{i} \right\rfloor = 0 \;\; \forall\, i \in I°(p)$ then p+2 ∈ P

Since p+2 is not divisible by any odd number between 3 and p and at the same time it's not divisible by any even number since p+2 is odd. ∎

Theorem1

Let $n \in N, n > 1$, and let p the last prime lower than or equal to n such that (p, p+2) is a pair of primes, then:

$$\pi(n) = \sum_2^p \left(\left\{ \frac{p+2}{\bar{p}_i} \right\} - \left\{ \frac{p+1}{\bar{p}_i} \right\} \right) \quad \textit{for each prime } \bar{p}_i \leq p \leq n \in N \qquad (3)$$

<u>*Proof*</u>

First part:

We establish that $\left\{ \frac{p+2}{i} \right\} - \left\{ \frac{p+1}{i} \right\} = 1 \quad \forall\, i \in I°(p) \quad (4)$

Let *p* ∈ P and consider a generic term of the summation (1) indicating the necessary and sufficient condition for having *p, p+2* ∈ P.

Since each addend is greater than or equal to zero, eq. (1) becomes:

$$\left\lfloor \frac{p+2}{i} \right\rfloor - \left\lfloor \frac{p+1}{i} \right\rfloor = 0 \quad \forall\, i \in I°(p) \; (5)$$

We observe that:

$p+2=k_1\, i + q_1$ con $i > q_1 > 0$

$p+1=k_2\, i + q_2$ con $i > q_2 > 0$

Since it must be $k_1 = k_2$ (eq. 5), it follows that for any $p \in P$ such that $p+2 \in P$ eq. (5) leads to:

$q_2 - q_1 = 1$ *i.e.* $\left\{\frac{p+2}{i}\right\} - \left\{\frac{p+1}{i}\right\} = 1$ (5b)

We have established that for $p \in P$ then $p+2 \in P$ if and only if the difference of fractional parts (4) calculated for any odd number between 3 and n, is equal to 1.

Second part:

Eq. (5) and eq. (4), can be restricted to the primes $\bar{p} \in I°(p)$, *i.e.* to the primes between *3* and *p*.

Proof

If $\sum_i \left(\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor\right) = 0$ for each odd $i \in I°(p)$

Obviously the same holds for each prime $\bar{p} \in I°(p)$:

$$\left\lfloor\frac{p+2}{\bar{p}}\right\rfloor - \left\lfloor\frac{p+1}{\bar{p}}\right\rfloor = 0$$

At the same time:

Let $\bar{p} \in I°(p)$, if $\left\lfloor\frac{p+2}{\bar{p}}\right\rfloor - \left\lfloor\frac{p+1}{\bar{p}}\right\rfloor = 0$ (6)

Each of his multiple lower than or equal to *p* verifies the same condition:

$$\left\lfloor\frac{p+2}{\bar{p}\, n}\right\rfloor - \left\lfloor\frac{p+1}{\bar{p}\, n}\right\rfloor = 0$$

In fact:

$\bar{k}_1 = \frac{p+2-\bar{q}_1}{n\cdot\bar{p}}$ and $\bar{k}_2 = \frac{p+1-\bar{q}_2}{n\cdot\bar{p}}$

$\bar{k}_1 = \frac{k_1 \cdot \bar{p} + q_1 - \bar{q}_1}{n\cdot\bar{p}}$ and $\bar{k}_2 = \frac{k_2 \cdot \bar{p} + q_2 - \bar{q}_2}{n\cdot\bar{p}}$

Hence:

$\bar{k}_1 = \frac{k_1 \cdot \bar{p}}{n\cdot\bar{p}} + \frac{q_1 - \bar{q}_1}{n\cdot\bar{p}}$ e $\bar{k}_2 = \frac{k_2 \cdot \bar{p}}{n\cdot\bar{p}} + \frac{q_2 - \bar{q}_2}{n\cdot\bar{p}}$

But $k_1 = k_2$ (by assumption (6))

hence $q_1 - \bar{q}_1 = q_2 - \bar{q}_2$

and since $q_1 - q_2 = 1$ from eq. (5b) it follows that: $\bar{q}_1 - \bar{q}_2 = 1$

Hence, if $k_1 = k_2$ then $\bar{k}_1 = \bar{k}_2$

As a consequence, if it is verified eq. (6), the same condition holds for each number $i \in I°(p)$.

We have established that for p ∈ P, necessary and sufficient condition for p+2 ∈ P is:

$$\left\{\frac{p+2}{\bar{p}}\right\} - \left\{\frac{p+1}{\bar{p}}\right\} = 1 \quad \text{for each prime } \bar{p} \le p \quad (7)$$

Third Part:

The case with $\bar{p}_i = 2$ is a consequence of the demonstration of lemma 1:

Since number 2 doesn't divide *p+2* (odd number):

$$\left\lfloor\frac{p+2}{2}\right\rfloor - \left\lfloor\frac{p+1}{2}\right\rfloor = 0$$

Hence the demonstration of Th.1 first part, leads to:

$$\left\{\frac{p+2}{2}\right\} - \left\{\frac{p+1}{2}\right\} = 1$$

We have established that for ∈ P such that p+2 ∈ P the difference of the fractional parts (7) calculated for each prime lower than or equal to p, is always equal to 1.

It follows that the summation of the fractional parts (7) 'counts' exactly the number of primes lower than or equal to a given number n ∈ *N* n > 1.

In other words theorem1 establishes that the number of primes lower than or equal to a given number n ∈ *N* n > 1 is the summation of the difference (7) of the fractional parts of *p+2* ∈ *P e p+1* for each prime lower than or equal to n. ■

Example

For n=30, the last pair of twin primes is (29 31)

	Difference of fractional parts (7)
1	0
2	1
3	1
5	1
7	1
9	1
11	1
13	1
15	1
17	1
19	1
21	1
23	1
25	1
27	1
29	1

It's evident that counting the primes lower than or equal to n = 30 is counting the fractional part according to (7).

Theorem2: *Twin pairs are infinitely many.*

Proof

Let's consider the limit:

$$\lim_{n\to+\infty} \pi(p_n) = \lim_{n\to+\infty} \sum_2^{p_n} \left(\left\{\frac{p_n+2}{\bar{p}_i}\right\} - \left\{\frac{p_n+1}{\bar{p}_i}\right\} \right) \qquad \text{with } p_n, p_n + 2 \in P \text{ and } \bar{p}_i \le p_n \le n$$

For the divergence of the first side of the equation, we have:

$$\lim_{n\to+\infty} \sum_2^{p_n} \left(\left\{\frac{p_n+2}{\bar{p}_i}\right\} - \left\{\frac{p_n+1}{\bar{p}_i}\right\} \right) = +\infty \qquad (8)$$

But considering eq. (7), it is possible if only there exist infinitely many twin primes.

In fact let's suppose (*reductio ad absurdum*) that the pairs of twin primes are finite in number and let $\dot{p}$ the last prime such that $(\dot{p}, \dot{p} + 2)$ is a pair of primes.

In this case by definition of fractional part function:

$$\left\{\frac{\acute{p} + 2}{\bar{p}_i}\right\} = \frac{\acute{p} + 2}{\bar{p}_i} - \left\lfloor\frac{\acute{p} + 2}{\bar{p}_i}\right\rfloor$$

But: $\lim_{n\to+\infty} \bar{p}_i = \lim_{n\to+\infty} p_n = +\infty$

In fact p_n is a monotonically increasing sequence hence the limit exists and the limit is not finite since in this case for each number $n \in N$ we have $p_n < M \in N$.

But this is in contradiction to Tchebycheff's[1] theorem. Hence $\lim_{n\to+\infty} p_n = +\infty$.

Hence: $\lim_{n\to+\infty} \left\{\frac{\acute{p}+2}{\bar{p}_i}\right\} = \lim_{n\to+\infty} \frac{\acute{p}+2}{\bar{p}_i} - \lim_{n\to+\infty} \left\lfloor\frac{\acute{p}+2}{\bar{p}_i}\right\rfloor = 0$

The same procedure applied to *p+1* leads to:

$$\lim_{n\to+\infty} \left\{\frac{\acute{p}+1}{\bar{p}_i}\right\} = \lim_{n\to+\infty} \frac{\acute{p}+1}{\bar{p}_i} - \lim_{n\to+\infty} \left\lfloor\frac{\acute{p}+1}{\bar{p}_i}\right\rfloor = 0$$

Hence we have a result in contradiction to the assumption that the pairs of twin primes are finite in number. The statement of the theorem follows. ■

[1] Bertrand-Tchebychev's theorem /**11**/ statements that for every integer n > 1 there is always at least one prime *p* such that: $n < p < 2n$

On the basis of theorem1, it is easy to demonstrate a new characterization of twin primes:

Corollary1

Be p a prime, $p > 7$, *necessary and sufficient condition for* $p + 2 \in P$ *is:*

$$\prod_{3}^{\lfloor p/3 \rfloor} \left\{ \frac{p+2}{\bar{p}_i} \right\} \neq 0 \qquad (9)$$

Proof

If p+2 $\in P$ then $\left\{\frac{p+2}{\bar{p}_i}\right\} \neq 0 \quad \forall \bar{p}_i \leq p$ and eq.(9) is proved.

At the same time if $\left\{\frac{p+2}{\bar{p}_i}\right\} \neq 0 \quad \forall \; \bar{p}_i$ between 3 and $\left\lfloor\frac{p}{3}\right\rfloor$ then:

p+2 = k $\bar{p}$ + q with $\bar{p} > q > 0$

Then:

p+1 = k $\bar{p}$ + q-1 with q-1 ≥ 0

Hence $\left\lfloor\frac{p+2}{\bar{p}_i}\right\rfloor = \left\lfloor\frac{p+1}{\bar{p}_i}\right\rfloor \forall \bar{p}_i$ between 3 and $\left\lfloor\frac{p}{3}\right\rfloor$

And from the theorem of S.M. Ruiz[2] we have that $p + 2 \in P$ *i.e.* p, $p + 2 \in P$. ■

[2] If i = 1 we have: $\left\lfloor\frac{p}{i}\right\rfloor - \left\lfloor\frac{p-1}{i}\right\rfloor = 2$ hence if $i \geq 3$ $\sum_{i\, odd.}^{j}\left(\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor\right) = 0 \quad j \leq \lfloor p/3 \rfloor$

The statement of the Corollary1 leads to Corollary2:

A generalization to other prime *k*-tuples[3]

Let p a prime, necessary and sufficient condition for $\{p, p+i\ , p+j, \dots, p+k\}\ \in P$ *is:*

$$\prod_{2}^{\left\lfloor\frac{n+k}{3}\right\rfloor}\left\{\frac{n\cdot(n+i)\cdot(n+j)\ldots.(n+k)}{p_i}\right\} \neq 0 \quad with\ \ 2 \leq\ p_i \leq \left\lfloor\frac{n+k}{3}\right\rfloor$$

<u>*Proof*</u>

From Corollary 1, we have[4]:

$$\prod_{2}^{\left\lfloor\frac{n+k}{3}\right\rfloor}\left\{\frac{n}{p_i}\right\}\cdot\left\{\frac{n+i}{p_i}\right\}\cdot\left\{\frac{n+j}{p_i}\right\}\cdot\ \ldots\cdot\left\{\frac{n+k}{p_i}\right\} \neq 0 \quad with\ \ 2 \leq\ p_i \leq \left\lfloor\frac{n+k}{3}\right\rfloor \qquad (10)$$

If $\{p, p+i\ , p+j, \dots, p+k\}$ are not divisible by p_i then also the product:

$n\cdot(n+i)\cdot(n+j)\ldots.(n+k)$ is not divisible by p_i.

At the same time if $n\cdot(n+i)\cdot(n+j)\ldots.(n+k)$ then:

$p, p+i\ , p+j, \dots, p+k$ are not divisible by p_i.

The statement of corollary 2 follows. ■

[3] A finite collection of values with a repeatable pattern of differences between primes
[4] The summation starts with $p_i = 2$ to erase even numbers. This is not necessary in eq. (9) since *p, p+2* are odd.

Example # 1

Let's consider $p = 29$ with $p+2 = 31 \in P$ where $\left\lfloor \frac{n}{3} \right\rfloor = 9$ i.e. $\bar{p}_i = 2,3,5,7$

$\left\{\frac{p+2}{3}\right\} = 1$ $\left\{\frac{p+2}{5}\right\} = 1$ $\left\{\frac{p+2}{7}\right\} = 3$

$$\prod_{3}^{\lfloor p/3 \rfloor} \left\{\frac{p+2}{\bar{p}_i}\right\} \neq 0$$

Let's now consider $p = 31$ with $p+2 = 33 \notin P$ where $\left\lfloor \frac{n}{3} \right\rfloor = 9$ i.e. $\bar{p}_i = 2,3,5,7$

$\left\{\frac{p+2}{3}\right\} = 0$ $\left\{\frac{p+2}{5}\right\} = 3$ $\left\{\frac{p+2}{7}\right\} = 5$

In this case:

$$\prod_{3}^{\lfloor p/3 \rfloor} \left\{\frac{p+2}{\bar{p}_i}\right\} = 0$$

Example # 2

Let's consider $\{11, 13, 17, 19\}$ where: $n + k = 19$ and $\left\lfloor \frac{n+k}{3} \right\rfloor = 6$ *i.e.* $\bar{p}_i = 2,3,5$

$n \cdot (n+i) \cdot (n+j) \ldots . (n+k) = 4689$

From eq. 10:

$$\prod_{3}^{\left\lfloor \frac{n+k}{3} \right\rfloor} \left\{\frac{n \cdot (n+i) \cdot (n+j) \ldots . (n+k)}{p_i}\right\} = 4$$

Now let's consider $\{11, 13, 16, 19\}$ where: $n + k = 19$ and $\left\lfloor \frac{n+k}{3} \right\rfloor = 6$ i.e. $p_i = 2,3,5$

$n \cdot (n+i) \cdot (n+j) \ldots . (n+k) = 4372$

From eq. 10:

$$\prod_{3}^{\left\lfloor \frac{n+k}{3} \right\rfloor} \left\{\frac{n \cdot (n+i) \cdot (n+j) \ldots . (n+k)}{p_i}\right\} = 0$$

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